Asymptotically analytic and other plurisubharmonic singularities

Alexander Rashkovskii

Abstract

We study classes of singularities of plurisubharmonic functions that can be approximated by analytic singularities with control over the residual Monge-Ampère masses. They are characterized in terms of Demailly's approximations, graded families of ideals of analytic functions and the corresponding asymptotic multiplier ideals, as well as in terms of relative types. The types relative to such singularities are represented as lower envelopes of weighted divisorial valuations and certain disk functionals. Finally, for functions with asymptotically analytic singularities we prove a Minkowski's type inequality for the residual Monge-Ampère masses and a plurisubharmonic variant of Musţată's summation formula for multiplier ideals.

Subject classification: 32U05, 32U25, 32U35, 14B05

Key words: plurisubharmonic singularity, analytic singularity, Demailly's approximation, Monge-Ampère mass, multiplier ideal

1 Introduction

Let u be a plurisubharmonic function on a complex manifold X of dimension $n > 1$. We are interested in asymptotic behavior of u near its *singularity point* x (where u takes the $-\infty$ value). Loosely speaking, this asymptotic behavior is what we call the *singularity* of u at x; more precisely, a singularity is an equivalence set with respect to the relation $u \sim v \Leftrightarrow u = v + O(1)$ near x.

Already in dimension 1, the asymptotic behavior of a subharmonic function can be quite complicated, however everything becomes easy when restricting to the case of functions that are harmonic in a punctured neighborhood of x, namely, $u(z) = \nu \log|z - x| + O(1)$ near x, where $\nu = \frac{1}{2\pi}\Delta u(\{x\})$.

In several variables, this corresponds to consideration of plurisubharmonic functions u that are *maximal* outside x; when u is locally bounded outside x, this means that it satisfies the homogeneous Monge-Ampère equation in a punctured neighborhood of x. Nevertheless, even in this class the variety in plurisubharmonic singularities is enormous. For instance, their collection contains those of functions $u = a \log|F| + O(1)$ for holomorphic mappings F from a neighborhood of x to $\mathbb{C}^N$,

$N \geq 1$. To some extent, we can consider such *analytic* singularities as "simple" objects and try to use them for approximation of arbitrary plurisubharmonic singularities. Note that although the passage to maximal singularities can change the asymptotic, it keeps its standard characteristics such as Lelong number, residual Monge-Ampère mass, integrability index, etc. In this sense, we lose little when replacing a plurisubharmonic function by the corresponding swept out maximal function.

There are some indications that an analytic approximation would work. First, it is the classical Lelong–Bremermann theorem on uniform approximation of continuous plurisubharmonic functions by functions of the form $\max_i a_i \log |f_i|$. Second, the celebrated Approximation Theorem due to Demailly [3] states that for any plurisubharmonic function u there exists a sequence of functions $\mathfrak{D}_k u$ with analytic singularities, converging to u pointwise and in L^1_{loc}; moreover, these functions keep track on the singularity of u – for example, their Lelong numbers converge to the Lelong number of u. On the other hand, assuming x to be an isolated singularity point of u (such functions are called *weights*), it is not clear if the residual Monge-Ampère masses $(dd^c\mathfrak{D}_k u)^n(\{x\})$ of $\mathfrak{D}_k u$ converge to that of u, even if u is maximal outside x. This is definitely not so (for any analytic approximation!) if u has zero Lelong number at x and positive mass there, because then $\mathfrak{D}_k u$ are locally bounded near x and hence of zero residual mass; existence of such a function u is however a well known open problem. Since the Monge-Ampère mass is a crucially important characteristic of plurisubharmonic singularity (for instance, $(dd^c \log |F|)^n(\{x\})$ is the multiplicity of the mapping F at x), this uncertainty is rather regretful.

It turns out that the convergence of the masses of $\mathfrak{D}_k u$ is inseparably linked with good analytic approximability of u. In the present note, we study a few classes of plurisubharmonic singularities that are well-suited for analytic approximations. To start with, we introduce a metric ρ on the collection of isolated maximal singularities, based on the notion of *relative type* [17] $\sigma(u, \varphi) = \liminf_{z \to x} u(z)/\varphi(z)$, and we say that a singularity ψ is *asymptotically analytic* if it belongs to the ρ-closure of the set of analytic singularities. Asymptotical analyticity of a weight ψ means actually that for any $\epsilon > 0$ one can find an analytic weight ψ_ϵ such that $(1+\epsilon)\psi_\epsilon + O(1) \leq \psi \leq (1-\epsilon)\psi_\epsilon + O(1)$. Examples of such singularities are exponentially Hölder continuous functions and those depending only on the absolute values of the variables. On the other hand, we have no examples of maximal singularities that are not asymptotically analytic.

We obtain several characterization of such singularities. Some of them use the

notion of generalized Green functions with respect to plurisubharmonic singularities [22]: given a maximal singularity φ at a point x in a bounded hyperconvex domain D, the corresponding Green function G_φ is the upper envelope of negative plurisubharmonic functions u in D such that $\sigma(u,\varphi)\ge 1$. Since $G_\varphi=\varphi+O(1)$, it depicts the same singularity as φ. We will use the Green functions as a kind of uniformization for families of the singularities.

First, we test the Green functions for the Demailly approximations $\mathfrak{D}_k\psi$ of a maximal weight ψ. In spite of the convergence of $\mathfrak{D}_k\psi$ to ψ, there is no direct reason for their Green functions to converge to G_ψ. Moreover, this again faces the aforementioned problem on zero Lelong number and nonzero Monge-Ampère mass. We prove that a subsequence of the Green functions of $\mathfrak{D}_k\psi$ – namely, $G_{\mathfrak{D}_{m!}\psi}$ – decreases to the function $\widehat{G}_\psi=\inf_k G_{\mathfrak{D}_k\phi}\ge G_\psi$; the proof is based on the Subadditivity Theorem for multiplier ideals [6]. We show that $\widehat{G}_\psi$ can be viewed as an "analytic greenification" of ψ, that is, the upper envelope of all negative plurisubharmonic functions v in D such that $\sigma(v,\varphi)\ge\sigma(\psi,\varphi)$ for all analytic weights φ. It is proved to coincide with G_ψ if and only if ψ can be approximated from *above* by analytic singularities whose residual Monge-Ampère masses converge to the mass of ψ; we call such singularities *inf analytic*, and they form our other main class of singularities that are close to analytic. Any asymptotical analytic singularity is inf analytic, and a stronger convergence describes asymptotical analyticity: $G_{\mathfrak{D}_k\psi}/G_\psi\to 1$ uniformly on $D\setminus\{x\}$; it implies, in particular, $(dd^c\mathfrak{D}_k\psi)^n(\{x\})\to(dd^c\psi)^n(\{x\})$ and $\exp G_\psi\in C(D)$.

Another characteristic property of asymptotically analytic weights makes use of the notion of asymptotic multiplier ideals [7]. We show first that, given a graded family $\mathfrak{a}_\bullet$ of primary ideals $\mathfrak{a}_k\subset\mathcal{O}_x$, their Green functions $G_{\mathfrak{a}_k}$ [19] converge, after a rescaling, to a weight $G_{\mathfrak{a}_\bullet}$, while the Green functions $G_{\mathfrak{j}_k}$ of the corresponding asymptotic multiplier ideals $\mathfrak{j}_k$ converge (also after rescaling) to a weight $G_{\mathfrak{j}_\bullet}\ge G_{\mathfrak{a}_\bullet}$. The multiplicities $e(\mathfrak{a}_\bullet)$ and $e(\mathfrak{j}_\bullet)$ (in the sense of [14]) are just the Monge-Ampère masses of $G_{\mathfrak{a}_\bullet}$ and $G_{\mathfrak{j}_\bullet}$, respectively, and they coincide if and only if the limit functions are equal.

The equality is then proved for the case of families generated by "good" (in particular, asymptotically analytic) singularities. Specifically, we consider $\mathfrak{a}_k$ defined by the conditions $\sigma(\log|f|,\psi)\ge k$. The upper regularization of $\sup_k h_k$, where $h_k=k^{-1}G_{\mathfrak{a}_k}$, is shown to coincide with G_ψ if and only if ψ can be approximated from *below* by analytic singularities whose residual Monge-Ampère masses converge to the mass of ψ, and for such a singularity, $h_k\to G_\psi$ in $L^n(D)$. The weight ψ is

asymptotically analytic if and only if $h_k/G_\psi \to 1$ uniformly on $D \setminus \{x\}$.

The lower envelope $\inf_k H_k$ of the functions $H_k = k^{-1}G_{\mathfrak{j}_k}$ equals G_ψ when ψ has analytic approximations from both below and above, and, if this is the case, H_k converge to G_ψ in $L^n(D)$ as well. When ψ is asymptotically analytic, we have again $H_k/h_k \to 1$ uniformly on $D \setminus \{x\}$.

Previously, the equality $e(\mathfrak{a}_\bullet) = e(\mathfrak{j}_\bullet)$ was known in few situations – for instance, when $\mathfrak{a}_k$ are determined by Abhyankar valuations [8] (which can be interpreted as a very special case of the weight ψ; in return, a stronger result is proved there), or when they are monomial [14]. A new feature of our approach is that we show what the "limits" of $\mathfrak{a}_\bullet$ and $\mathfrak{j}_\bullet$ – or, more precisely, the scaled limits of their logarithmic images $\log|\mathfrak{a}_\bullet|$ and $\log|\mathfrak{j}_\bullet|$ – are: they are the collections of plurisubharmonic functions u satisfying $\sigma(u, G_{\mathfrak{a}_\bullet}) \geq 1$ and $\sigma(u, G_{\mathfrak{j}_\bullet}) \geq 1$, respectively.

Another portion of results concerns types relative to asymptotically analytic and inf analytic weights. While the residual Monge-Ampère masses of Demailly's approximations characterize inf analytic weights, asymptotical analyticity can be completely described in terms of relative types. Namely, $\sigma(\mathfrak{D}_k u, \psi) \to \sigma(u, \psi)$ for every u if and only if ψ is asymptotically analytic. Since the Lelong number is the type relative to the weight $\psi(z) = \log|z - x|$, this gives a stronger property of Demailly's approximations than that on convergence of their Lelong numbers.

In addition, the type of u relative to an inf analytic weight ψ turns out to coincide with the infimum of weighted divisorial valuations arising as generic Lelong numbers of pullbacks μ^*u of u along exceptional primes of proper modifications $\mu : \hat{X} \to X$ over x. As a consequence, it can be represented as the lower envelope of the disk functionals

$$\rho_\gamma(u, \psi) = \liminf_{\zeta \to 0} \frac{\gamma^* u(\zeta)}{\gamma^* \psi(\zeta)},$$

where γ are analytic maps from the unit disk to X such that $\gamma(0) = x$. When both u and ψ have analytic singularities, this was proved (by algebraic methods) in [12]; note that in commutative algebra, the object corresponding to the relative type is known as Samuel's asymptotic function. For *tame* weights, a slightly smaller class of plurisubharmonic weights than asymptotically analytic ones, these representations were essentially proved in [1]; we believe however that our methods are more elementary.

The obtained characterizations of asymptotically/inf analytic weights show limits of using Demailly's approximations in describing plurisubharmonic singularities – unless all maximal singularities are such, which is doubtful (although this remains

an open question).

We conclude with two results that perhaps are true not only for asymptotic analytic singularities, however we do not know how to prove them in the general case; our proofs just use the corresponding statements from algebraic geometry and the good approximability of asymptotical analytic singularities. The first result extends an inequality due to Teissier from [9] to the plurisubharmonic situation, giving thus $\tau_{u+v}^{1/n} \le \tau_u^{1/n} + \tau_v^{1/n}$, where τ_w is the residual Monge-Ampère mass of w at x. The second statement concerns the multiplier ideal of the function $\max\{u, v\}$ and is based on a result from [15].

2 Preliminaries

For basics on plurisubharmonic functions and the complex Monge-Ampère operator, we refer the reader to [10]. In this section, we recall some notions of particular importance for us and set the corresponding notation.

2.1 Plurisubharmonic singularities

Let X be a complex manifold of dimension n, and let PSH_x be the collection of germs of plurisubharmonic functions at $x \in X$. We will say that a plurisubharmonic function u is *singular* at x if u is not bounded (below) in any neighborhood of x.

The equivalence class $\mathrm{cl}(u)$ of $u \in \mathrm{PSH}_x$ with respect to the relation "$u \sim v$ if $u(z) = v(z) + O(1)$" will be called the *plurisubharmonic singularity* of u; in [22], a closely related object was introduced under the name "standard singularity".

Germs with isolated singularities at x (i.e., that are locally bounded outside x) will be occasionally called *weights* and their collection will be denoted by $\mathrm{W_x}$. The Monge-Ampère operator $(dd^c)^n$ is well defined on such functions; the *residual mass* of $\varphi \in \mathrm{W_x}$ at x is

$$\tau_\varphi = (dd^c u)^n(\{x\}).$$

A weight $\varphi \in \mathrm{W_x}$ is called *maximal* if it is a maximal plurisubharmonic function on a punctured neighborhood of x (satisfies $(dd^c\varphi)^n = 0$ outside x). A basic example is $\varphi = \log|F|$ for an equidimensional holomorphic mapping F with isolated zero at x, and in this case τ_φ equals the multiplicity of F. The collection of all nonconstant maximal weights will be denoted by $\mathrm{MW_x}$, and $\mathrm{cl}(\mathrm{MW_x})$ is the collection of all *maximal singularities*.

2.2 Relative types and Lelong numbers

For any function $u \in \mathrm{PSH}_x$, we denote its *type relative to a weight* $\varphi \in \mathrm{MW_x}$ [17] by

$$\sigma(u,\varphi) = \liminf_{z\to x} \frac{u(z)}{\varphi(z)} = \lim_{r\to-\infty} r^{-1}\Lambda(u,\varphi,r), \tag{2.1}$$

where

$$\Lambda(u,\varphi,r) = \sup\{u(z) : \varphi(z) < r\}. \tag{2.2}$$

It is a finite nonnegative number, and maximality of φ implies the bound

$$u \le \sigma(u,\varphi)\varphi + O(1). \tag{2.3}$$

Evidently, $\sigma(u,\varphi) = \sigma(v,\psi)$ for any $u \in \mathrm{cl}(u)$, $\psi \in \mathrm{cl}(\varphi)$, so the relative type is a function on singularities.

If f is an analytic function and F is an equidimensional holomorphic mapping, then, as follows from [12], $\sigma(\log|f|, \log|F|)$ equals Samuel's asymptotic function [20]

$$\bar\nu_I(f) = \lim_{k\to\infty} k^{-1}\max\{m\in\mathbb{Z}_+ :\ f^k \in I^m\}$$

for the ideal I generated by the components of the mapping F.

When $\varphi(z) = \log|\zeta(z) - \zeta(x)|$, where ζ are local coordinates near x, $\sigma(u,\varphi)$ is the Lelong number $\nu(u,x)$ of u at x. Since only local behavior of plurisubharmonic functions is considered here, we will omit the indication on coordinate functions and use the denotation $|z-x|$.

Given an irreducible analytic variety $M \subset X$, the value

$$\nu(u,M) = \inf\{\nu(u,y) :\ y \in M\} \tag{2.4}$$

is the *generic Lelong number of* u *along* M. By Siu's theorem, $\nu(u,x) = \nu(u,M)$ for all $x \in M\setminus M'$, where M' is a proper analytic subset of M.

2.3 Green and Green-like functions

We will use the following extremal function introduced by V. Zahariuta [22], see also [23]. Let $D \subset X$ be a bounded hyperconvex domain; $PSH^-(D)$ will mean the collection of all negative plurisubharmonic functions on D.

Given a weight $\varphi \in \mathrm{MW_x}$, $x \in D$, denote

$$G_\varphi(z) = G_{\varphi,D}(z) = \sup\{u(z) :\ u \in PSH^-(D),\ \sigma(u,\varphi) \ge 1\}.$$

The function is plurisubharmonic in D, maximal in $D\setminus\{x\}$, $G_\varphi \in \mathrm{cl}(\varphi)$, and $G_\varphi = 0$ on ∂D; moreover, it is a unique function with these properties. Furthermore, if φ is continuous near x, then G_φ is continuous on D. (The continuity of $\phi \in \mathrm{W_x}$ is understood here as continuity of $\exp\phi$.) We will refer to this function as the *Green* (or *Green–Zahariuta*) *function with singularity* φ.

If $\varphi(z) = \log|z-x|$, then G_φ is the standard pluricomplex Green function $G_{x,D}$ of D with pole at x.

Any function $u \in \mathrm{PSH}_x$ generates various maximal functions that can be viewed as its *greenifications*. First, let P be an arbitrary collection of maximal weights; consider the family $\mathcal{M}^P_{u,D} = \{v \in PSH^-(D) :\ \sigma(v,\varphi) \geq \sigma(u,\varphi) \quad \forall \varphi \in P\}$; the function

$$h^P_u(z) = h^P_{u,D}(z) = \sup\{v(z) :\ v \in \mathcal{M}^P_{u,D}\} \tag{2.5}$$

is called the *type-greenification* of u with respect to the collection P [17]. For $\phi \in \mathrm{W_x}$, the function $h^P_\phi \in \mathrm{MW_x}$, equals 0 on ∂D and satisfies $\sigma(h^P_\phi, \varphi) = \sigma(\phi,\varphi)$ for all $\varphi \in P$; it can also be represented as the best plurisubharmonic minorant of the family $\{G_\psi :\ \psi \in P,\ \sigma(\phi,\psi) \geq 1\}$ [17, Prop. 5.1, 5.2]. The *raison d'être* for such a function is that it gives the best possible bound on $u \in \mathrm{PSH}_x$, $u \leq h^P_u + O(1)$, when the only available information on u is the values $\sigma(u,\varphi)$ for $\varphi \in P$.

Alternatively, we can consider the class $\mathcal{F}_{\phi,D}$ of negative plurisubharmonic functions v in D such that $v(z) \leq \phi(z) + O(1)$ near x, then the regularization of its upper envelope,

$$g_\phi(z) = g_{\phi,D}(z) = \limsup_{y\to z} \sup\{v(y) :\ v \in \mathcal{F}_{\phi,D}\}, \tag{2.6}$$

is a plurisubharmonic function in D, the *complete greenification* of ϕ at x in D [17]. It is maximal on $D\setminus\{x\}$ and equals zero on ∂D. If $\varphi \in \mathrm{MW_x}$, then $g_\varphi = G_\varphi$, the Green–Zahariuta function for φ. When φ is not maximal, one need not have the equality $\varphi = g_\varphi + O(1)$; nevertheless, the relative types and the residual Monge-Ampère mass remain the same [17, Prop. 5.5, 5.6]: $\sigma(\varphi,\psi) = \sigma(g_\varphi,\psi)$ for every $\psi \in \mathrm{MW_x}$ and

$$\tau_\varphi = \tau_{g_\varphi}. \tag{2.7}$$

The greenifications are related by $g_\phi = h^{\mathrm{MW_x}}_\phi \leq h^{P_1}_\phi \leq h^{P_2}_\phi$ for any $\phi \in \mathrm{W_x}$ and $P_2 \subset P_1 \subset \mathrm{MW_x}$.

2.4 Multiplier ideals and Demailly's approximation

Given $u \in \mathrm{PSH}(X)$, let $\mathcal{J}(u)$ denote the *multiplier ideal* for u, that is, the ideal of functions $f \in \mathcal{O}(X)$ such that $|f|e^{-u} \in L^2_{loc}$. A fundamental property of special importance for us is the Subadditivity Theorem [6, Thm. 2.6]

$$\mathcal{J}(u+v) \subseteq \mathcal{J}(u) \cdot \mathcal{J}(v) \quad \forall u, v \in \mathrm{PSH}(X). \tag{2.8}$$

We refer to [4], [11] for detailed information on multiplier ideals and their applications to analysis and algebraic geometry.

The notion was used by Demailly for constructing approximations of plurisubharmonic functions by ones with analytic singularities. For any $D \Subset X$, the restriction of $\mathcal{J}(u)$ to D is generated as $\mathcal{O}_D$-module by any basis of the Hilbert space

$$\mathcal{H}(u) = \{f \in \mathcal{O}(D) : \ |f|\, e^{-u} \in L^2(D)\}.$$

Demailly's Approximation Theorem [3] says that any function $u \in \mathrm{PSH}_x$ can be approximated, in a neighborhood D of x, by the functions

$$\mathfrak{D}_k u = \frac{1}{k} \sup\{\log|f| : \ f \in \mathcal{O}(D), \ \int_D |f|^2 e^{-2ku}\, dV < 1\} = \frac{2}{k} \log \sum_i |f_{k,i}|^2, \tag{2.9}$$

where $\{f_{k,i}\}_i$ is an orthonormal basis for $\mathcal{H}(ku)$, in a way that

$$u(z) - \frac{C}{k} \le \mathfrak{D}_k u(z) \le \sup_{|\zeta - z|<r} u(\zeta) + \frac{1}{k} \log \frac{C}{r^n}, \quad z \in D. \tag{2.10}$$

Since there exist finitely many functions $f_{k,i}$, $1 \le i \le i_0 = i_0(ku)$, such that

$$\mathfrak{D}_k u(z) = \frac{2}{k} \log \sum_{i \le i_0} |f_{k,i}(z)|^2 + O(1), \quad z \to x,$$

all the functions $\mathfrak{D}_k u$ have analytic singularities. As a consequence of (2.10), they converge to u pointwise and in L^1_{loc}. In addition, the Lelong numbers $\nu(\mathfrak{D}_k u, x)$ of the functions $\mathfrak{D}_k$ at x tend to the Lelong number $\nu(u, x)$.

3 Asymptotically analytic singularities

We will say that $u \in \mathrm{PSH}_x$ has *analytic singularity* if $c \log|F| \in \mathrm{cl}(u)$ for some $c > 0$ and a holomorphic mapping F of a neighborhood of x to $\mathbb{C}^N$.

If a weight $\phi \in \mathrm{W_x}$ has analytic singularity, then the mapping F has isolated zero at x. As is known, the integral closure of the ideal generated by the components F_j

of F has precisely n generators $\xi_k = \sum a_{k,j} F_j$ (generic linear combinations of F_j), $k = 1, \dots, n$, so $\log |F| = \log |\xi| + O(1)$. Therefore, ϕ is equivalent to a maximal weight. The collection of all weights with analytic singularities at x will be denoted by $\mathrm{AW_x}$.

We are going to deal with singularities that are "close" to analytic ones, in the sense of the following metric.

Proposition 3.1 *The function* $\rho : \mathrm{MW_x} \times \mathrm{MW_x} \to [0, \infty]$ *given by*

$$\rho(\varphi, \psi) = -\log \min\{\sigma(\varphi, \psi), \sigma(\psi, \varphi)\}$$

is a metric on maximal singularities $\mathrm{cl}(\mathrm{MW_x})$.

Proof. As follows from the definition of relative types,

$$\sigma(\varphi, \psi) \geq \sigma(\varphi, \eta)\sigma(\eta, \psi) \tag{3.1}$$

for all $\varphi, \eta, \psi \in \mathrm{MW_x}$. In particular, $\min\{\sigma(\varphi, \psi), \sigma(\psi, \varphi)\} \leq 1$ with an equality if and only if $\varphi = \psi + O(1)$, which follows from (2.3). Therefore, $\rho \geq 0$ and it equals zero only when $\mathrm{cl}(\varphi) = \mathrm{cl}(\psi)$. The symmetry of ρ is evident, and (3.1) implies the triangle inequality. □

We will say that $\psi \in \mathrm{MW_x}$ has *asymptotically analytic singularity* if it belongs to the closure (in the sense of the metric ρ) of $\mathrm{AW_x}$. The collection of all such weights will be denoted by $\mathrm{AAW_x}$.

Proposition 3.2 *A weight* $\psi \in \mathrm{MW_x}$ *has asymptotically analytic singularity if and only if for every* $\epsilon > 0$ *there exists a weight* $\psi_\epsilon \in \mathrm{AW_x}$ *such that*

$$(1 + \epsilon)\psi_\epsilon + O(1) \leq \psi \leq (1 - \epsilon)\psi_\epsilon + O(1). \tag{3.2}$$

Proof. Relations (3.2) are equivalent to $\sigma(\psi_\epsilon, \psi) \geq (1 + \epsilon)^{-1}$ and $\sigma(\psi, \psi_\epsilon) \geq 1 - \epsilon$ and thus imply $\psi \in \mathrm{AAW_x}$. Conversely, assuming the asymptotical analyticity, we can find weights $\psi_\epsilon \in \mathrm{AW_x}$ such that

$$\min\{\sigma(\varphi, \psi), \sigma(\psi, \varphi)\} \geq \max\{1 - \epsilon, (1 + \epsilon)^{-1}\},$$

which implies (3.2). □

Relations (3.2) do not use the maximality of ψ and can be thus used as a definition of asymptotical analyticity for non-maximal weights (and even for arbitrary plurisubharmonic functions).

Unlike analytic weights, an asymptotically analytic weight need not be equivalent to a maximal one. Take, for example, $\psi = \log|z| - |\log|z||^{1/2}$ in the unit ball, then $(1+\epsilon)\log|z| - C_\epsilon \le \psi \le \log|z|$, while $g_\psi = \log|z|$. On the other hand, there is always a weight $g_\psi \in \mathrm{AAW_x}$ such that $\lim \psi/g_\psi = 1$, which follows from

Proposition 3.3 *Let $\psi \in \mathrm{W_x}$ be a weight with asymptotically analytic singularity and let D be a hyperconvex neighborhood of the point x. Then there exists the limit*

$$\lim_{z\to x} \frac{\psi(z)}{g_\psi(z)} = 1,$$

where g_ψ is the complete greenification of ψ (2.6) in D. As a consequence, for any $u \in \mathrm{PSH}_x$ one can set

$$\sigma(u,\psi) = \limsup_{z\to x} \frac{u(z)}{\psi(z)} = \sigma(u, g_\psi).$$

Proof. Given $\epsilon \in (0,1)$, choose a weight $\psi_\epsilon \in \mathrm{AW_x}$ and a constant $C_\epsilon > 0$ such that $(1+\epsilon)\psi_\epsilon - C_\epsilon \le \psi \le (1-\epsilon)\psi_\epsilon + C_\epsilon$. Then $(1+\epsilon)G_{\psi_\epsilon} \le g_\psi \le (1-\epsilon)G_{\psi_\epsilon}$, where G_{ψ_ϵ} is the Green function for ψ_ϵ in D, and thus

$$\frac{1-3\epsilon}{1+\epsilon} \le \frac{\psi(z)}{g_\psi} \le \frac{1+4\epsilon}{1-\epsilon}$$

for all z such that

$$1-\epsilon < \frac{\psi_\epsilon(z)}{G_{\psi_\epsilon}(z)} < 1+\epsilon$$

(recall that $\psi_\epsilon = G_{\psi_\epsilon} + O(1)$) and $G_{\psi_\epsilon}(z) < -\epsilon^{-1}C_\epsilon$. $\square$

Remark. Without the asymptotic analyticity assumption the result need not be true (even in dimension 1, just take $\psi \in \mathrm{W}_0$ such that $\limsup_{z\to 0}\psi(z)/\log|z| = \infty$).

Example 3.4 According to [1], a continuous weight $\varphi \in \mathrm{W_x}$ is called *tame* if there exists a constant $C > 0$ such that for every $t > C$ and every analytic germ f from the multiplier ideal $\mathcal{J}(t\varphi)$ of $t\varphi$ at x (that is, the function $fe^{-t\varphi}$ is L^2-integrable near x), one has $\log|f| \le (t-C)\varphi + O(1)$. For maximal weights φ, the latter can be written as $\sigma(\log|f|,\varphi) \ge t - C$. Demailly's approximations $\mathfrak{D}_k\varphi$ (2.9) of a tame weight φ satisfy, by [1, Lemma 5.9],

$$\varphi + O(1) \le \mathfrak{D}_k\varphi \le (1 - C_\varphi/k)\varphi + O(1) \tag{3.3}$$

near x; moreover, conditions (3.3) characterize tame weights. Therefore, all tame weights have asymptotically analytic singularity.

Example 3.5 In particular, any exponentially Hölder continuous weight φ, that is, satisfying

$$e^{\varphi(y)} - e^{\varphi(z)} \le |y-z|^{\beta} \quad \text{near } x, \tag{3.4}$$

is tame [1, Lemma 5.10], which deduces from Demailly's approximation theorem (2.10). As a consequence, all weights with analytic singularities are tame.

Example 3.6 Let $x = 0 \in X \subset \mathbb{C}^n$. Any multicircular weight $\varphi \in \mathrm{MW}_0$ (depending on $|z_1|, \ldots, |z_n|$ only) has asymptotically analytic singularity, which can be shown as follows. First of all, such a function φ is equivalent to its *indicator at* 0, that is the Green function for φ in the unit polydisk $\mathbb{D}^n$ [13], so we assume $\varphi = G_\varphi$. This implies $\varphi(A_c z) = c\,\varphi(z)$ for every $c > 0$, where

$$A_c(z) = (|z_1|^c, \ldots, |z_n|^c), \quad z \in \mathbb{D}^n.$$

Then the function $\tilde\varphi(t) := \varphi(e^{t_1}, \ldots, e^{t_n})$ is convex and positive homogeneous in $\mathbb{R}^n_-$, equal to zero on $\partial\mathbb{R}^n_-$, and thus represents as

$$\tilde\varphi(t) = \sup\{\langle a, t\rangle : a \in \Gamma\}, \quad t \in \mathbb{R}^n_-,$$

where $\Gamma \subset \mathbb{R}^n_+$. Therefore, for any $\epsilon > 0$ there exist positive integers m and N and monomials $z^{k(j)}$, $k(j) \in \mathbb{Z}^n_+$, $1 \le j \le m$, such that $|\varphi(z) - \varphi_\epsilon(z)| < \epsilon/2$ for all z with $-1 \le \varphi(z) \le 0$, where

$$\varphi_\epsilon(z) = N^{-1} \max_j \log|z^{k(j)}|.$$

Take any $w \in \mathbb{D}^n$ with $\varphi(w) = t < -1$ and let $z = A_c w$ with $c = 1/|2t|$. Then $\varphi(z) = -1/2$, $A_1 w = A_{|2t|} z$, and

$$|\varphi(w) - \varphi_\epsilon(w)| = |\varphi(A_1 w) - \varphi_\epsilon(A_1 w)| = |\varphi(A_{|2t|} z) - \psi_\epsilon(A_{|2t|} z)| < |t|\epsilon.$$

Since $\varphi(w) = t$, this implies

$$(1+\epsilon)\varphi(w) \le \varphi_\epsilon(w) < (1-\epsilon)\varphi(w)$$

for all w with $\varphi(w) < -1$.

Remark. We have no example of $\varphi \in \mathrm{MW_x} \setminus \mathrm{AAW_x}$.

4 Convergence of Green functions

In what follows, we fix a bounded hyperconvex domain $D \subset X$ containing x. Given $\phi \in \mathrm{MW_x}$, let G_ϕ denote the Green function of D for the singularity ϕ. Since any analytic weight is equivalent to a maximal weight, the Green functions $G_{\mathfrak{D}_k\phi}$ for Demailly's approximations $\mathfrak{D}_k\phi$ (2.9) of ϕ are well defined, too.

4.1 Green functions of Demailly's approximations

By Demailly's approximation theorem, the functions $\mathfrak{D}_k\phi$ converge to ϕ in L^1_{loc} for any $\phi \in \mathrm{W_x}$. We do not know if this implies convergence of the Green functions $G_{\mathfrak{D}_k\phi}$ to G_ϕ. This is certainly not so if ϕ has zero Lelong number (because then $G_{\mathfrak{D}_k\phi} \equiv 0$), however existence of such a maximal weight ϕ is equivalent to a famous open problem on plurisubharmonic weights with zero Lelong number and positive residual Monge-Ampère mass.

We can show that there is a subsequence of the Green functions decreasing to the lower envelope $\widehat{G}_\phi$ of the functions $G_{\mathfrak{D}_k\phi}$ (and thus $\widehat{G}_\phi$ is plurisubharmonic).

Proposition 4.1 *If $\phi \in \mathrm{MW_x}$, then the Green functions $G_{\mathfrak{D}_{m!}\phi}$ for the singularities $\mathfrak{D}_{m!}\phi$ decrease to the function $\widehat{G}_\phi = \inf_k G_{\mathfrak{D}_k\phi} \in \mathrm{MW_x}$.*

Proof. Let $\mathcal{J}(k\phi)$ be the multiplier ideals for the functions $k\phi$, $k \in \mathbb{Z}_+$. By the Subadditivity Theorem (2.8), $\mathcal{J}((mk)\phi) \subseteq \mathcal{J}(k\phi)^m$ for all $m, k \in \mathbb{Z}_+$. Since the multiplier ideals $\mathcal{J}(k\phi)$ generate Demailly's approximations $\mathfrak{D}_k\phi$, we have $\mathfrak{D}_{mk}\phi \le \mathfrak{D}_k\phi + C(k,m)$ and so,

$$G_{\mathfrak{D}_{mk}\phi} \le G_{\mathfrak{D}_k\phi}, \tag{4.1}$$

which implies that the sequence $G_{\mathfrak{D}_{m!}\phi}$ is decreasing to some function $\widehat{G}_\phi$, plurisubharmonic in D and maximal in $D \setminus \{x\}$. Moreover, (4.1) yields

$$G_{\mathfrak{D}_{m!}\phi} \le G_{\mathfrak{D}_k\phi} \quad \forall k \le m,$$

so

$$\widehat{G}_\phi = \inf_k G_{\mathfrak{D}_k\phi} = \lim_{m\to\infty} G_{\mathfrak{D}_{m!}\phi} \ge G_\phi.$$

□

The function $\widehat{G}_\phi$ has a nice interpretation in terms of greenifications. Namely, let h^A_ϕ denote the type-greenification (2.5) with respect to the family $\mathrm{AW_x}$:

$$h^A_\phi = \sup\{v \in \mathrm{PSH}^-(D) : \ \sigma(v,\varphi) \ge \sigma(\phi,\varphi) \quad \forall \varphi \in \mathrm{AW_x}\}. \tag{4.2}$$

Proposition 4.2 *$\widehat{G}_\phi = h^A_\phi$ for any $\phi \in \mathrm{W_x}$. As a consequence, there exists a sequence $\varphi_j \in \mathrm{AW_x}$ such that G_{φ_j} decrease to h^A_ϕ.*

Proof. As was mentioned in Section 2.3, the function h^A_ϕ is the best plurisubharmonic minorant of the family $\{G_\psi : \ \psi \in \mathrm{AW_x}, \ \sigma(\phi,\psi) \ge 1\}$. Since all the functions $G_{\mathfrak{D}_k\phi}$ belong to that family, $h^A_\phi \le G_{\mathfrak{D}_k\phi}$ and thus $h^A_\phi \le \widehat{G}_\phi$. On the other hand, the

relation $\sigma(\phi,\psi)\ge 1$ means $\phi\le\psi+O(1)$ and thus implies $\mathfrak{D}_k\phi\le\mathfrak{D}_k\psi+O(1)$, so $G_{\mathfrak{D}_k\psi}\ge G_{\mathfrak{D}_k\phi}\ge\widehat{G}_\phi$ for all k. As $\psi\in\mathrm{AW_x}$, relation (3.3) implies $G_{\mathfrak{D}_k\psi}\to G_\psi$, which gives $G_\psi\ge\widehat{G}_\phi$ for all $\psi\in\mathrm{AW_x}$ with $\sigma(\phi,\psi)\ge 1$, so $h_\phi^A\ge\widehat{G}_\phi$. Finally, the sequence $\varphi_j=\mathfrak{D}_{j!}\,\phi$ provides a sequence of Green functions decreasing to h_ϕ^A. $\square$

Proposition 4.2 means that the singularity of $\widehat{G}_\phi$ is the best upper bound on the singularity of ϕ when the latter can be tested on all analytic weights. This gives us one more motivation on the problem if $\widehat{G}_\phi$ coincides with G_ϕ.

It turns out that the relation $\widehat{G}_\phi=G_\phi$ is completely controlled by the behavior of the Monge-Ampère masses of the functions $\mathfrak{D}_k\phi$.

Proposition 4.3 *If $\phi\in\mathrm{MW_x}$, then $\widehat{G}_\phi=G_\phi$ if and only if $\sup_k\tau_{\mathfrak{D}_k\phi}=\tau_\phi$.*

Proof. This follows from Proposition 4.1 and Lemma 4.4 below. $\square$

Lemma 4.4 *Let $\varphi_j,\varphi\in\mathrm{PSH}^-(D)$ be locally bounded and maximal on $D\setminus\{x\}$ and equal to 0 on ∂D.*

(i) If $\varphi_j\ge\varphi$ decrease to a function ψ, then $\psi=\varphi$ if and only if $\tau_{\varphi_j}\to\tau_\varphi$;

(ii) If $\varphi_j\le\varphi$ increase to a function η, then $\eta^*=\varphi$ if and only if $\tau_{\varphi_j}\to\tau_\varphi$.

Proof. (i) Since the Monge-Ampère operator is continuous with respect to decreasing sequences, the function ψ is maximal on $D\setminus\{x\}$ and $\tau_{\varphi_j}\to\tau_\psi$. Assuming $\tau_\psi=\tau_\varphi$, we get the two functions, $\psi\ge\varphi$, maximal on $D\setminus\{x\}$, equal to 0 on ∂D, and with the same residual Monge-Ampère mass at x. By Lemma 4.5 below, $\psi=\varphi$.

(ii) Similar proof, the relation $\tau_{\varphi_j}\to\tau_{\eta^*}$ in this case being due to Bedford–Taylor's result for increasing sequences of bounded plurisubharmonic functions, applied to the functions $\psi_j=\max\{\varphi_j,-1\}$, since $\tau_{\varphi_j}=(dd^c\psi_j)^n(D)$. (Alternatively, one can refer to the convergence results for increasing sequences in more general classes of plurisubharmonic functions [2], [21].) $\square$

Lemma 4.5 [17, Lemma 6.3] *Let D be a bounded hyperconvex domain and let $\phi_1,\phi_2\in\mathrm{PSH}(D)\cap L^\infty_{loc}(D\setminus\{x\})$ be two functions such that $(dd^c\phi_i)^n=\delta_x$ and $\phi_i|_{\partial D}=0$. If $\phi_1\ge\phi_2$ in D, then $\phi_1=\phi_2$.*

If ϕ is a tame weight, then (3.3) implies

$$G_\phi\le G_{\mathfrak{D}_k\phi}\le(1-C_\phi/k)G_\phi \tag{4.3}$$

and, therefore, uniform convergence of $G_{\mathfrak{D}_k\phi}/G_\phi$ to 1. We will prove that the convergence holds true for any weight $\phi\in\mathrm{AAW_x}$ as well. In particular, this will give $\widehat{G}_\phi=G_\phi$ for such weights.

Theorem 4.6 *Given $\psi \in \mathrm{MW_x}$, let G_ψ and $G_{\mathfrak{D}_k\psi}$ be the Green functions of a bounded hyperconvex domain D with the singularities ψ and $\mathfrak{D}_k\psi$, respectively. Then*

$$\frac{G_{\mathfrak{D}_k\psi}}{G_\psi} \to 1 \text{ uniformly on } D\setminus\{x\} \tag{4.4}$$

if and only if $\psi \in \mathrm{AAW_x}$. For such a weight ψ, we get thus $\tau_{\mathfrak{D}_k\psi} \to \tau_\psi$ and $\widehat{G}_\psi = G_\psi$.

Proof. Relation (4.4) implies (3.2) with $\psi_\epsilon = \mathfrak{D}_k\psi$ and $k \geq k(\epsilon)$.

Let us prove the reverse implication. For $\epsilon > 0$, let $\psi_\epsilon \in \mathrm{AW_x}$ be a weight from (3.2). Then $(1+\epsilon)G_{\psi_\epsilon} \leq G_\psi \leq G_{\mathfrak{D}_k\psi} \leq G_{\mathfrak{D}_k(1-\epsilon)\psi_\epsilon}$ for all k. Therefore,

$$\frac{G_{\mathfrak{D}_k(1-\epsilon)\psi_\epsilon}(z)}{(1+\epsilon)G_{\psi_\epsilon}(z)} \leq \frac{G_{\mathfrak{D}_k\psi}(z)}{G_\psi(z)} \leq 1, \quad z \in D\setminus\{x\}. \tag{4.5}$$

Since $(1-\epsilon)\psi_\epsilon$ has analytic singularity, (4.3) implies

$$G_{\mathfrak{D}_k(1-\epsilon)\psi_\epsilon} \leq (1 - C_\epsilon/k)(1-\epsilon)G_{\psi_\epsilon}$$

and so, (4.5) for all $k \geq k(\epsilon)$ gives us

$$1 - 2\epsilon \leq \frac{G_{\mathfrak{D}_k\psi}(z)}{G_\psi(z)} \leq 1, \quad z \in D\setminus\{x\},$$

which gives us (4.4) and, by Demailly's Comparison Theorem, the convergence of $\tau_{\mathfrak{D}_k\psi}$ to τ_ψ. □

Corollary 4.7 *The Green function G_ϕ of a bounded hyperconvex domain D with singularity $\phi \in \mathrm{AAW_x}$ is continuous on D.*

So, asymptotical analyticity is rather a strong property with regard to behavior of the singularities of $\mathfrak{D}_k\phi$. In order to get the largest class of weights ϕ such that $\widehat{G}_\phi = G_\phi$, we will say that a weight $\phi \in \mathrm{MW_x}$ has *inf analytic singularity* if there exists a sequence of weights $\phi_j \in \mathrm{AW_x}$ such that $\phi \leq \phi_j + O(1)$ and $\tau_{\phi_j} \to \tau_\phi$. The class of these weights will be denoted by $\mathrm{IAW_x}$; evidently, $\mathrm{IAW_x} \supseteq \mathrm{AAW_x}$.

Theorem 4.8 *Let $\phi \in \mathrm{MW_x}$, then $\phi \in \mathrm{IAW_x}$ if and only if $G_\phi = \inf_k G_{\mathfrak{D}_k\phi}$.*

Proof. Let $\phi \in \mathrm{IAW_x}$. Then for any $m, j \in \mathbb{Z}_+$ we have $G_{\mathfrak{D}_{m!}\phi} \leq G_{\mathfrak{D}_{m!}\phi_j}$, which implies $\tau_{\mathfrak{D}_{m!}\phi} \geq \tau_{\mathfrak{D}_{m!}\phi_j}$. Given $\epsilon > 0$, choose j such that $\tau_{\phi_j} > \tau_\varphi - \epsilon/2$. Since $\phi_j \in \mathrm{AW_x}$, we can then choose, by Theorem 4.6, m_0 such that $\tau_{\mathfrak{D}_{m!}\phi_j} \geq \tau_{\phi_j} - \epsilon/2$ for all $m \geq m_0$. Therefore, $\tau_{\mathfrak{D}_{m!}\phi} \geq \tau_\phi - \epsilon$ for all $m \geq m_0$, and Propositions 4.1 and 4.3 gives us $\widehat{G}_\phi = G_\phi$. The converse is evident by means of the choice $\phi_j = \mathfrak{D}_{j!}\phi$. □

In view of Proposition 4.2, this can be restated as follows.

Corollary 4.9 *A weight $\phi \in \mathrm{MW_x}$ belongs to $\mathrm{IAW_x}$ if and only if $G_\phi = h_\phi^A$ (4.2).*

Remark. We do not know if $G_{\mathfrak{D}_k\phi} \to G_\phi$ for any $\phi \in \mathrm{IAW_x}$.

4.2 Asymptotic multiplier ideals

Another nice property of asymptotically analytic weights comes from consideration of asymptotic multiplier ideals, the notion introduced in [7]. Recall that a family $\mathfrak{a}_\bullet$ of ideals $\mathfrak{a}_k \subset \mathcal{O}_x$ is called *graded* if $\mathfrak{a}_m \cdot \mathfrak{a}_k \subseteq \mathfrak{a}_{mk}$ for all positive integers m and k; we assume $\mathfrak{a}_k \neq \{0\}$ for $k >> 0$. If all $\mathfrak{a}_k$ are primary (zero dimensional), then there exists the limit

$$e(\mathfrak{a}_\bullet) = \lim_{k\to\infty} k^{-n} e(\mathfrak{a}_k) \tag{4.6}$$

called the *multiplicity* of $\mathfrak{a}_\bullet$ (here $e(\mathfrak{a}_k)$ are the Samuel multiplicities of $\mathfrak{a}_k$) [14, Cor. 1.5]. As follows from the Subadditivity Theorem, the corresponding family of multiplier ideals $\{\mathcal{J}(\mathfrak{a}_p^{k/p})\}_{p\in\mathbb{N}}$ has a unique maximal element, denoted here $\mathfrak{j}_k$, and it coincides with $\mathcal{J}(\mathfrak{a}_p^{k/p})$ for all sufficiently great values of p, see [7]. The ideals $\mathfrak{j}_\bullet = \{\mathfrak{j}_k\}$ are called *asymptotic multiplier ideals* of $\mathfrak{a}_\bullet$. One has always

$$\mathfrak{j}_{km} \subseteq \mathfrak{j}_k^m \tag{4.7}$$

and

$$\mathfrak{a}_k \subseteq \mathfrak{j}_k, \tag{4.8}$$

see [7, Prop. 1.7]; furthermore, as shown in [14], there exists the limit

$$e(\mathfrak{j}_\bullet) = \lim_{k\to\infty} k^{-n} e(\mathfrak{j}_k). \tag{4.9}$$

In some cases the ideals $\mathfrak{j}_\bullet$ are not much bigger than $\mathfrak{a}_\bullet$ in the sense $e(\mathfrak{a}_\bullet) = e(\mathfrak{j}_\bullet)$ (for instance, when $\mathfrak{a}_k$ are defined by Abhyankar valuations [8] or when they are monomial [14]).

Let us apply a machinery of Green functions in order to deal with such families of ideals. Given a bounded hyperconvex neighborhood D of x, let $G_{\mathfrak{a}_k}$ denote the Green function of D with singularity along $\mathfrak{a}_k$ [19], i.e. with the singularity $\varphi_k = \log|F_k|$, where F_k are holomorphic mappings whose component generate and $\mathfrak{a}_k$, and let $h_k = k^{-1} G_{\mathfrak{a}_k}$. Similarly, we denote $H_k = k^{-1} G_{\mathfrak{j}_k}$; as follows from (4.8),

$$H_k \geq h_k. \tag{4.10}$$

Since the family $\mathfrak{a}_\bullet$ is graded, we have $\mathfrak{a}_k^m \subseteq \mathfrak{a}_{km}$ and thus $h_k \leq h_{km}$, hence we can argue as in the proof of Proposition 4.1. In doing so, we derive that the

sequence $h_{m!}$ is increasing to the function $h_{\mathfrak{a}_\bullet} := \sup_k h_k$. Let us denote its upper regularization $(h_{\mathfrak{a}_\bullet})^*$ by $G_{\mathfrak{a}_\bullet}$.

In the same manner, relation (4.7) implies $H_k \geq H_{km}$, so the sequence $H_{m!}$ is decreasing to the function $G_{\mathfrak{j}_\bullet} := \inf_k H_k \in \mathrm{MW_x}$. By (4.10),

$$G_{\mathfrak{a}_1} \leq G_{\mathfrak{a}_\bullet} \leq G_{\mathfrak{j}_\bullet} \leq G_{\mathfrak{j}_1}, \tag{4.11}$$

so $G_{\mathfrak{a}_\bullet}, G_{\mathfrak{j}_\bullet} \in \mathrm{MW_x}$.

To show that the monotone convergence of $h_{m!}$ and $H_{m!}$ implies a convergence of all h_k and H_k, we need the following

Lemma 4.10 *Let $u, v \in \mathrm{PSH}^-(D)$ be maximal on $D \setminus \{x\}$, equal to 0 on ∂D, and $u \leq v$ in D. Then*

$$\int_D (v-u)^n dd^c w_1 \wedge \ldots \wedge dd^c w_n \leq n! \int_D w_1 \left[(dd^c v)^n - (dd^c u)^n\right]$$

for all $w_j \in \mathrm{PSH}(D)$, $-1 \leq w_j \leq 0$.

Proof. This is a particular case of [16, Prop. 3.4]. □

Proposition 4.11 *(i) $h_k \to G_{\mathfrak{a}_\bullet}$ and $H_k \to G_{\mathfrak{j}_\bullet}$ in $L^n(D)$;*

(ii) $G_{\mathfrak{a}_\bullet} = G_{\mathfrak{j}_\bullet}$ if and only if $e(\mathfrak{a}_\bullet) = e(\mathfrak{j}_\bullet)$.

Proof. *(i)* Note first that, since the Samuel multiplicity $e(I)$ of a primary ideal I generated by $f_1, \ldots, f_m$, equals the residual Monge-Ampère masse of the function $\log|f|$ [5, Lemma 2.1], we have $e(\mathfrak{a}_k) = k^n \tau_{h_k}$ and $e(\mathfrak{j}_k) = k^n \tau_{H_k}$. Since $h_{m!}$ increase to $h_{\mathfrak{a}_\bullet}$ and $H_{m!}$ decrease to $G_{\mathfrak{j}_\bullet}$, we have $\tau_{h_{m!}} \to \tau_{\mathfrak{a}_\bullet}$ and $\tau_{H_{m!}} \to \tau_{\mathfrak{j}_\bullet}$, so (4.6) and (4.9) give us $\tau_{h_k} \to \tau_{\mathfrak{a}_\bullet}$ and $\tau_{H_k} \to \tau_{\mathfrak{j}_\bullet}$. In addition, $h_k \leq G_{\mathfrak{a}_\bullet}$ and $H_k \geq G_{\mathfrak{j}_\bullet}$ and all these functions are maximal on $D \setminus \{x\}$, equal zero on ∂D. Therefore, the statement follows from Lemma 4.10.

(ii) follows from (4.11) by Lemma 4.5. □

We can use results from Section 4.1 to find conditions for the equality $G_{\mathfrak{a}_\bullet} = G_{\mathfrak{j}_\bullet}$. Since $\mathfrak{j}_k = \mathcal{J}(\mathfrak{a}_p^{k/p}) = \mathcal{J}(kh_p)$ for $p > p(k)$, we have $H_k = \mathfrak{D}_k h_p + O(1)$ for $p > p(k)$. Therefore,

$$H_k \leq \mathfrak{D}_k G_{\mathfrak{a}_\bullet} + O(1).$$

In view of Theorem 4.8, this gives us the following

Proposition 4.12 $G_{\mathfrak{a}_\bullet} \leq H_k \leq G_{\mathfrak{D}_k G_{\mathfrak{a}_\bullet}}$. *Therefore, if $G_{\mathfrak{a}_\bullet} \in \mathrm{IAW_x}$, then $G_{\mathfrak{a}_\bullet} = G_{\mathfrak{j}_\bullet}$.*

Let us now specify

$$\mathfrak{a}_k = \mathfrak{a}_k(\phi) = \{f \in \mathcal{O}_x : \sigma(\log|f|, \phi) \ge k\}, \quad \phi \in \mathrm{MW_x}.$$

Since $\sigma(u+v,\phi) \ge \sigma(u,\phi) + \sigma(v,\phi)$, it is a graded family of ideals. To provide they are different from $\{0\}$ and are primary, we assume that the weight ϕ has finite Łojasiewicz exponent, i.e.

$$\limsup_{z\to x} \frac{\phi(z)}{\log|z-x|} < \infty.$$

We will see that, for such a choice of $\mathfrak{a}_k$, the relation $G_{\mathfrak{a}_\bullet} = G_{\mathfrak{j}_\bullet} = G_\phi$ is true for asymptotically analytic and even more general singularities ϕ.

Since $\sigma(h_k, \phi) \ge 1$, we have $h_k \le G_\phi$ and thus

$$G_{\mathfrak{a}_\bullet} \le G_\phi. \tag{4.12}$$

We will say that a weight $\phi \in \mathrm{MW_x}$ has *sup-analytic singularity* if $\phi \ge \phi_j + O(1)$ for some weights $\phi_j \in \mathrm{AW_x}$ with $\tau_{\phi_j} \to \tau_\phi$; the collection of such weights is denoted by $\mathrm{SAW_x}$. Again, $\mathrm{SAW_x} \supseteq \mathrm{AAW_x}$. Note also that any weight with sup-analytic singularity has finite Łojasiewicz exponent.

Proposition 4.13 $G_\phi = G_{\mathfrak{a}_\bullet}$ *if and only if* $\phi \in \mathrm{SAW_x}$.

Proof. Let $\phi \in \mathrm{SAW_x}$. In view of (4.12), we need to show only that $G_\phi \le G_{\mathfrak{a}_\bullet}$.

Notice that the functions $\psi_j = \max\{\phi_i : i \le j\}$ have analytic singularities and satisfy $\phi \ge \psi_j + O(1)$ and $\tau_\phi \le \tau_{\psi_j} \le \tau_{\phi_j}$, so $\tau_{\psi_j} \to \tau_\phi$. In addition, the corresponding Green functions G_{ψ_j} increase to some function $\widehat{\psi}$. By Lemma 4.4, $\widehat{\psi}^* = G_\phi$.

Denote $\mathfrak{a}_k^{(j)} = \{f \in \mathcal{O}_x : \sigma(\log|f|, \psi_j) \ge k\}$, then $\mathfrak{a}_k^{(j)} \subseteq \mathfrak{a}_k$ and hence $h_k^{(j)} \le h_k$ and $G_{\mathfrak{a}_\bullet}^{(j)} \le G_{\mathfrak{a}_\bullet}$. Since ϕ_j have analytic singularity, $G_{\mathfrak{a}_\bullet}^{(j)} = G_{\phi_j}$ and so, $G_{\mathfrak{a}_\bullet} \ge G_{\phi_j}$ for all j and thus $G_{\mathfrak{a}_\bullet} \ge G_\phi$.

The other direction is evident. □

Theorem 4.14 *Let the functions* $G_{\mathfrak{a}_\bullet}$ *and* $G_{\mathfrak{j}_\bullet}$ *be defined as above for a weight* $\phi \in \mathrm{IAW_x} \cap \mathrm{SAW_x}$. *Then* $G_{\mathfrak{a}_\bullet} = G_{\mathfrak{j}_\bullet} = G_\phi$ *and* $e(\mathfrak{a}_\bullet) = e(\mathfrak{j}_\bullet) = \tau_\phi$.

Proof. Proposition 4.13 implies $G_\phi = G_{\mathfrak{a}_\bullet}$. Then Proposition 4.12 becomes applicable and, in view of Proposition 4.11(ii), completes the proof. □

Remark. Since every Abhyankar valuation is generated by a relative type [1], Theorem 4.16 extends the mentioned result from [8] to arbitrary asymptotically

analytic weights (note however that for Abhyankar valuations a stronger result is proved there).

Since $\mathrm{AAW_x} \subseteq \mathrm{IAW_x} \cap \mathrm{SAW_x}$, the statement of Theorem 4.14 holds true for all asymptotically analytic singularities. It is not surprising that for such weights one can claim even more.

We start with a characterization of $\mathrm{AAW_x}$ in terms of the functions h_k.

Proposition 4.15 *Let a weight $\phi \in \mathrm{MW_x}$ have finite Łojasiewicz exponent, and let the functions h_k be defined as above. Then $\phi \in \mathrm{AAW_x}$ if and only if*

$$\frac{G_\phi}{h_k} \rightrightarrows 1 \quad \text{on } D \setminus \{x\}. \tag{4.13}$$

(Therefore, $G_\phi = \sup_k h_k = G_{\mathfrak{a}_\bullet}$ if $\phi \in \mathrm{AAW_x}$.)

Proof. Let ϕ have asymptotically analytic singularity, then for any $\epsilon \in (0,1)$ one can find a holomorphic mapping F_ϵ and a positive integer k such that

$$\frac{1}{k}\log|F_\epsilon| \le \phi + O(1) \le \frac{1-\epsilon}{k}\log|F_\epsilon|, \tag{4.14}$$

so all the components of F_ϵ belong to $\mathfrak{a}_k$ and thus $h_k \ge (1-\epsilon)^{-1}G_\phi$, which gives (4.13). Conversely, the inequality $G_\phi/h_k > (1-\epsilon)$ on $D \setminus \{x\}$ implies (4.14). $\square$

Theorem 4.16 *If the functions h_k, H_k, $G_{\mathfrak{a}_\bullet}$, and $G_{\mathfrak{j}_\bullet}$ are defined as above for a weight $\phi \in \mathrm{AAW_x}$, then $h_k/H_k \to 1$ uniformly on $D \setminus \{x\}$ and $G_{\mathfrak{a}_\bullet} = G_{\mathfrak{j}_\bullet} = G_\phi$.*

Proof. This follows from Propositions 4.15, 4.12 and Theorem 4.6. $\square$

Observe that our approach gives a more precise meaning to the fact that the families $\mathfrak{a}_\bullet$ and $\mathfrak{j}_\bullet$ are "close". Since all the ideals $\mathfrak{a}_k$ and $\mathfrak{j}_k$ are integrally closed, they are in a one-to-one correspondence with their Green functions $G_{\mathfrak{a}_k}$ and $G_{\mathfrak{j}_k}$ in the sense that, for example, $f \in \mathfrak{a}_k \Leftrightarrow \log|f| \le G_{\mathfrak{a}_k} + O(1)$. These functions have the scaled limits

$$G_{\mathfrak{a}_\bullet} = \lim_{k\to\infty} \frac{1}{k} G_{\mathfrak{a}_k}, \quad G_{\mathfrak{j}_\bullet} = \lim_{k\to\infty} \frac{1}{k} G_{\mathfrak{j}_k},$$

so

$$\lim_{k\to\infty} k^{-1}\log|\mathfrak{a}_k| = \{u \in \mathrm{PSH}_x : \sigma(u, G_{\mathfrak{a}_\bullet}) \ge 1\},$$

$$\lim_{k\to\infty} k^{-1}\log|\mathfrak{j}_k| = \{u \in \mathrm{PSH}_x : \sigma(u, G_{\mathfrak{j}_\bullet}) \ge 1\},$$

and for "good" weights ϕ, the limit objects equal $\{u \in \mathrm{PSH}_x : \sigma(u, \phi) \ge 1\}$.

5 Relative types

Let us notice the following simple properties of types relative to asymptotically analytic weights.

Proposition 5.1 *Let $\psi \in \mathrm{W_x}$ be an asymptotically analytic weight and let ψ_ϵ be analytic weights from (3.2), then $\sigma(u, \psi_\epsilon) \to \sigma(u, \psi)$ as $\epsilon \to 0$.*

Proof. By (3.2), $(1 - \epsilon)\sigma(u, \psi) \le \sigma(u, \psi_\epsilon) \le (1 + \epsilon)\sigma(u, \psi)$. □

Proposition 5.2 *If $\phi \in \mathrm{AAW_x}$, then $\sigma(u, \mathfrak{D}_k\phi) \to \sigma(u, \phi)$ for every $u \in \mathrm{PSH}_x$.*

Proof. This is a direct consequence of Theorem 4.6. □

The corresponding property of types relative to inf analytic singularities reads as follows.

Proposition 5.3 *If $\phi \in \mathrm{IAW_x}$, then $\sigma(u, \mathfrak{D}_{j!}\,\phi)$ decrease to $\sigma(u, \phi)$ for every $u \in \mathrm{PSH}_x$.*

Proof. Let us denote $\varphi_j = G_{\mathfrak{D}_{j!}\,\phi}$. By Proposition 4.2 and Corollary 4.9, the functions φ_j decrease to G_ϕ; therefore, $\sigma(u, \varphi_j)$ is a decreasing sequence for any $u \in \mathrm{PSH}_x$. We can set then

$$\sigma(u) = \lim_{j\to\infty} \sigma(u, \varphi_j).$$

We will show that $\sigma(u) = \sigma(u, \phi)$.

The functional $\sigma : \mathrm{PSH}_x \to [0, \infty]$ satisfies $\sigma(\max\{u, v\}) = \min\{\sigma(u), \sigma(v)\}$, is positive on $\log|z - x|$, finite on all $u \not\equiv -\infty$, positive homogeneous and lower semicontinuous because if $u_k \to u$ in L^1_{loc}, then for each j,

$$\limsup_{k\to\infty} \sigma(u_k) \le \limsup_{k\to\infty} \sigma(u_k, \varphi_j) \le \sigma(u, \varphi_j).$$

Therefore, by [17, Thm. 4.3], $\sigma(u) = \sigma(u, \psi)$ for some weight $\psi \in \mathrm{MW_x}$. Observe that $\sigma(u, \psi) \ge \sigma(u, \phi)$ for all u, so $G_\psi \ge G_\phi$. On the other hand, $\sigma(u, \psi) \le \sigma(u, \varphi_j)$ for all j and hence $G_\psi \le G_\phi$, which implies $\sigma(u, \psi) = \sigma(u, \phi)$. □

More advanced special properties of types relative to asymptotically/inf analytic weights are given below.

5.1 Types of Demailly's approximations

As stated in Demailly's Approximation Theorem, the Lelong numbers of $\mathfrak{D}_k u$ converge to that of u for every $u \in \mathrm{PSH}_x$. Here we extend this to types of $\mathfrak{D}_k u$ relative to asymptotical analytic weights.

We first relate the φ-types of the functions u_k to the φ-types of u for exponentially Hölder continuous weights φ.

Lemma 5.4 *Let $\varphi \in \mathrm{MW}_x$ satisfy (3.4) and let $u \in \mathrm{PSH}_x$. Then the types $\sigma(\mathfrak{D}_k u, \varphi)$ of its Demailly approximations $\mathfrak{D}_k u$ satisfy the relations*

$$\sigma(u,\varphi) - \frac{n}{k\beta} \le \sigma(\mathfrak{D}_k u, \varphi) \le \sigma(u,\varphi). \tag{5.1}$$

Proof. The first inequality in (2.10) implies the relation $\sigma(\mathfrak{D}_k u, \varphi) \le \sigma(u,\varphi)$.

If $\varphi(z) < r$ and $\log|z-\zeta| < r/\beta$, then (3.4) implies $\varphi(\zeta) \le r + \log 2$. Therefore, the second inequality in (2.10) with $\log\delta = r/\beta$ gives

$$\Lambda(\mathfrak{D}_k u, \varphi, r) \le \Lambda(u, \varphi, r + \log 2) - \frac{n}{k\beta} r + \frac{C_2}{k},$$

where the function Λ is defined by (2.2), and then (2.1) implies the first inequality in (5.1). □

This implies the convergence $\sigma(\mathfrak{D}_k u, \psi) \to \sigma(u,\psi)$ for $\psi \in \mathrm{AAW_x}$; moreover, this is one more characteristic property of asymptotically analytic weights.

Theorem 5.5 *A weight $\psi \in \mathrm{W_x}$ is asymptotically analytic if and only if*

$$\sigma(\mathfrak{D}_k u, \psi) \underset{k\to\infty}{\longrightarrow} \sigma(u,\psi) \quad \forall u \in \mathrm{PSH}_x. \tag{5.2}$$

Proof. That $\psi \in \mathrm{AAW_x}$ implies (5.2), this follows from (3.2), Proposition 5.1 and Lemma 5.4 because all the functions $\exp\psi_\epsilon$ are Hölder continuous (say, with exponents β_ϵ):

$$\frac{1}{1+\epsilon}\left(\sigma(u,\psi_\epsilon) - \frac{n}{k\beta_\epsilon}\right) \le \frac{\sigma(\mathfrak{D}_k u, \psi_\epsilon)}{1+\epsilon} \le \sigma(\mathfrak{D}_k u, \psi) \le \frac{\sigma(\mathfrak{D}_k u, \psi_\epsilon)}{1-\epsilon} \le \frac{\sigma(u,\psi_\epsilon)}{1-\epsilon}.$$

Conversely, relation (5.2) implies $\sigma(\mathfrak{D}_k\psi, \psi) \to 1$ and, since $\sigma(\psi, \mathfrak{D}_k\psi) \ge 1$, the convergence $\rho(\psi, \mathfrak{D}_k\psi) \to 0$. □

5.2 Representation by divisorial valuations

Let μ be a proper modification over a neighborhood of a point $x \in X$. For any function $f \in \mathcal{O}_x$, the generic multiplicity of $\mu^* f$ over an irreducible component of the exceptional divisor $\mu^{-1}(x)$ is a *divisorial*, or *Rees, valuation*. One can extend this notion to plurisubharmonic functions by replacing the multiplicity of $\mu^* f$ with the generic Lelong number of $\mu^* u$ over the component. Here we will represent types relative to inf analytic weights as envelopes of such valuations.

Let us take $\varphi = \log |F| \in MW_x$. By the Hironaka desingularization theorem, there exists a log resolution for the mapping F, i.e., a proper holomorphic mapping μ of a manifold $\hat{X}$ to a neighborhood U of x, that is an isomorphism between $\hat{X} \backslash \mu^{-1}(x)$ and $U \backslash \{x\}$, such that $\mu^{-1}(x)$ is a normal crossing divisor with components $E_1, \ldots, E_N$, and in local coordinates centered at a generic point p of a nonempty intersection $E_I = \cap_{i \in I} E_i$, where $I \subset \{1, \ldots, N\}$,

$$(F \circ \mu)(\hat{x}) = h(\hat{x}) \prod_{i \in I} \hat{x}_i^{m_i}$$

with $h(0) \neq 0$. Then for any $u \in \mathrm{PSH}_x$, one has

$$\sigma(u, \varphi) = \min\{\nu_{I,m_I}(\mu^* u) : E_I \neq \emptyset\}, \tag{5.3}$$

where

$$\nu_{I,m_I}(\mu^* u) = \liminf_{\hat{x} \to 0} \frac{(\mu^* u)(\hat{x})}{\sum_{i \in I} m_i \log |\hat{x}_i|}$$

at a generic point of $p \in E_I$.

We need the following elementary

Lemma 5.6 *Let $v(t)$ be a negative convex function on $\mathbb{R}^k_-$, $k > 1$, increasing in each t_i. Then*

$$\liminf_{t \to -\infty} \frac{v(t)}{\sum_i t_i} = \min_i \liminf_{t_i \to -\infty} \frac{v(t)}{t_i}. \tag{5.4}$$

Proof of Lemma 5.6. Denote the left hand side of (5.4) by A and

$$A_i = \liminf_{t_i \to -\infty} \frac{v(t)}{t_i}.$$

From the convexity of v it follows that for any point $t^* \in \mathbb{R}^k_-$ the ratio

$$\frac{v(t_1, t_2^*, \ldots, t_k^*) - v(t_1^*, t_2^*, \ldots, t_k^*)}{t_1 - t_1^*}$$

decreases when $t_1 \to -\infty$ and thus

$$v(t_1, t_2^*, \dots, t_k^*) - v(t_1^*, t_2^*, \dots, t_k^*) \le A_1(t_1 - t_1^*), \quad t_1 < t_1^*.$$

Similarly,

$$v(t_1, t_2, t_3^*, \dots, t_k^*) - v(t_1, t_2^*, t_3^*, \dots, t_k^*) \le A_2(t_2 - t_2^*), \quad t_2 < t_2^*,$$

and so on, the last inequality being

$$v(t_1, \dots, t_k) - v(t_1, \dots, t_{k-1}, t_k^*) \le A_k(t_k - t_k^*), \quad t_k < t_k^*.$$

Summing them up, we get

$$v(t) - v(t^*) \le \sum_i A_i(t_i - t_i^*) \le \min_i A_i \left(\sum_i t_i - \sum_i t_i^* \right),$$

which gives $A \ge \min_i A_i$. Since the reverse inequality is evident, the lemma is proved. □

From Lemma 5.6 applied to the function

$$v(t) = \sup\{\mu^* u(\hat{x}) :\ \log|\hat{x}_i| < m_i^{-1} t_i,\ i \in I\},$$

we deduce

$$\nu_{I,m_I}(\mu^* u) = \min_{i\in I} \nu_{i,m_i}(\mu^* u) = \min_{i \in I} m_i^{-1} \nu_{E_i}(\mu^* u),$$

where $\nu_{E_i}(\mu^* u)$ is the generic Lelong number of $\mu^* u$ along E_i (2.4). We denote this value by $\mathcal{R}_j(u)$ and call it the *divisorial valuation* of $u \in \mathrm{PSH}_x$ along E_i:

$$\mathcal{R}_j(u) = \nu_{E_i}(\mu^* u) = \inf\{\nu(\mu^* u, p) :\ p \in E_i\}.$$

Now (5.3) gives us the following result.

Theorem 5.7 *For any weight $\varphi = \log|F| \in \mathrm{AW_x}$ there exist finitely many divisorial valuations $\mathcal{R}_j$ and positive integers m_j such that*

$$\sigma(u, \varphi) = \min_j\ m_j^{-1} \mathcal{R}_j(u)$$

for every $u \in \mathrm{PSH}_x$.

Remark. For the case $u = \log|f|$ this follows also from [12, Thm. 4.1.6]. Conversely, one can deduce Theorem 5.7 from that result by applying Theorem 5.5.

Theorem 5.8 *If $\psi \in \mathrm{MW_x}$, then $\psi \in \mathrm{IAW_x}$ if and only if there exist denumerably many divisorial valuations $\mathcal{R}_j$ and positive numbers s_j such that*

$$\sigma(u,\psi) = \inf_j \, s_j \mathcal{R}_j(u) \quad \forall u \in \mathrm{PSH}_x. \tag{5.5}$$

Proof. Let $\psi \in \mathrm{IAW_x}$. By Theorem 5.7, for each j the type $\sigma(u, \mathfrak{D}_{j!}\,\psi)$ is the lower envelope of finitely many weighted divisorial valuations $s_j\mathcal{R}_j(u)$. Then the theorem follows from Proposition 5.3.

Conversely, (5.5) means that G_ψ is the best plurisubharmonic minorant of the function $\inf_j G_{\psi_j}$, where the weights ψ_j represent the valuations $s_j\mathcal{R}_j$ in the sense $\sigma(u,\psi_j) = s_j\mathcal{R}_j(u)$, see the proof of Proposition 5.3. All ψ_j are tame (and thus asymptotically analytic), and since the best plurisubharmonic minorant of the minimum of finitely many tame weights is tame as well, this means that there exists a sequence of weights φ_j with analytic singularities such that the Green functions G_{φ_j} decrease to G_ψ. Therefore, $\psi \in \mathrm{IAW_x}$. ☐

Remarks. 1. For asymptotically analytic weights, the arguments were briefly sketched in [18]. For tame weights, the representation follows from what was proved (by a different method) in [1].

2. Relative types have the obvious property $\sigma(\sum \alpha_j\, u_j, \varphi) \ge \sum \alpha_j\, \sigma(u_j, \varphi)$ and hence are concave functionals on PSH_x, while for the divisorial valuations $\mathcal{R}_j$ there is always an equality. From this point of view, relation (5.5) is similar to the representation of concave functions as lower envelope of linear ones, which holds on linear topological spaces. This can be put into the general picture of tropical analysis on plurisubharmonic singularities [17], [18].

5.3 Analytic disks

Another (although related) representation for the types can be given by means of analytic disks. To do so, we use arguments from [12]. Given $u \in \mathrm{PSH}_x$ and $\varphi \in \mathrm{MW_x}$, denote

$$\sigma^*(u,\varphi) = \inf_{\gamma\in\mathcal{A}_x} \liminf_{\zeta\to 0} \frac{\gamma^*u(\zeta)}{\gamma^*\varphi(\zeta)}, \tag{5.6}$$

where $\mathcal{A}_x$ is the collection of all analytic maps $\gamma : \mathbb{D} \to X$ such that $\gamma(0) = x$, and γ^*u is the pullback of u by γ.

Evidently, $\sigma^*(u,\varphi) \ge \sigma(u,\varphi)$ for any $\varphi \in \mathrm{MW_x}$. If f is a holomorphic function and F a holomorphic mapping with isolated zero at x, then $\sigma^*(\log|f|, \log|F|) =$

$\sigma(\log|f|, \log|F|)$ by [12, Thm. 5.2]; moreover, in this case there exists a curve $\gamma \in \mathcal{A}_x$ such that

$$\sigma(\log|f|, \log|F|) = \liminf_{\zeta\to 0} \frac{\log|\gamma^* f(\zeta)|}{\log|\gamma^* F(\zeta)|},$$

see [12, Prop. 5.4]. Similar arguments together with Theorems 5.7 and 5.8 give us the following result.

Theorem 5.9 *If $\psi \in \mathrm{IAW_x}$, then $\sigma^*(u,\psi) = \sigma(u,\psi)$ for every $u \in \mathrm{PSH}_x$. If, in addition, ψ has analytic singularity, then the infimum in (5.6) always attains.*

Proof. For the case of analytic weight $\psi = \log|F|$ we will use the idea from the proof of [12, Prop. 5.4]. By Theorem 5.7, $\sigma(u, \log|F|) = \min_j m_j^{-1}\mathcal{R}_j(u)$, where m_j and $\mathcal{R}_j(u)$ are generic Lelong numbers of $\log|\mu^*F|$ and μ^*u, respectively, along the exceptional primes E_j of a log resolution μ for the mapping F. Take a germ of an analytic curve $\hat\gamma_j \subset \hat X$ passing transversally through a point $\hat x_j \in E_j$ and such that the generic Lelong number of μ^*u along E_i equals the Lelong number of the restriction of μ^*u to $\hat\gamma_j$ at $\hat x_j$ (which is possible bye Siu's theorem). Then for the curve $\gamma_j = \mu^*\hat\gamma_j$ we get

$$\mathcal{R}_j(u) = \liminf_{\zeta\to 0} \frac{\gamma_j^* u\,(\zeta)}{\log|\gamma_j(\zeta) - x|}, \quad m_j = \lim_{\zeta\to 0} \frac{\log|\gamma_j^* F(\zeta)|}{\log|\gamma_j(\zeta) - x|},$$

so

$$\sigma(u, \log|F|) = \min_j \liminf_{\zeta\to 0} \frac{\gamma_j^* u\,(\zeta)}{\log|\gamma_j^* F(\zeta)|}.$$

For arbitrary $\psi \in \mathrm{IAW_x}$, we refer to Theorems 5.8. □

6 Other results

In this section we present some results that perhaps are not specific to functions with asymptotically analytic singularities, but we do not know how one can prove them without such an assumption.

6.1 Teissier's inequality

In an appendix to [9], B. Teissier proved the following Minkowski's type inequality. If I and J are two primary ideals in the ring $\mathcal{O}_x$, then

$$e(I\cdot J)^{1/n} \le e(I)^{1/n} + e(J)^{1/n}, \tag{6.1}$$

where $e(\mathcal{I})$ denotes the Samuel multiplicity of the ideal $\mathcal{I}$.

If an ideal $\mathcal{I}$ is generated by analytic functions $f_1, \ldots, f_m$, then $e(I) = \tau_{\log|f|}$, see [5]. Let us for any $u \in \mathrm{W_x}$ denote $\tilde{\tau}_u = \tau_u^{1/n}$.

Theorem 6.1 *If the complete greenifications (2.6) of $u, w \in \mathrm{W_x}$ have asymptotically analytic singularities, then*

$$\tilde{\tau}_{u+v} \le \tilde{\tau}_u + \tilde{\tau}_v.$$

Proof. In view of (2.7), we can assume $u, v \in \mathrm{AAW_x}$. Given $\epsilon > 0$, let $u_\epsilon = a_\epsilon \log |F_\epsilon|$ and $v_\epsilon = b_\epsilon \log |H_\epsilon|$ be analytic weights approximating u and v, respectively. We may assume that the factors a_ϵ and b_ϵ are rational, say, $a_\epsilon = p/N$ and $b_\epsilon = q/N$ (p, q, N depend on ϵ). Therefore,

$$u + v \ge (1+\epsilon) N^{-1} \log |F_\epsilon^p H_\epsilon^q| + O(1)$$

and thus

$$\tilde{\tau}_{u+v} \le (1+\epsilon)\tilde{\tau}_w$$

with $w = N^{-1} \log |F_\epsilon^p H_\epsilon^q|$.

Let I_ϵ and J_ϵ denote the ideals generated by the functions $\{F_{\epsilon,i}^p\}_j$ and $\{H_{\epsilon,i}^q\}_j$, respectively. Then $\tilde{\tau}_w = N^{-1} e(I_\epsilon \cdot J_\epsilon)^{1/n}$, and Teissier's inequality (6.1) implies

$$\tilde{\tau}_w \le N^{-1} \left[e(I_\epsilon)^{1/n} + e(J_\epsilon)^{1/n} \right] = \left[\tilde{\tau}_{u_\epsilon} + \tilde{\tau}_{v_\epsilon} \right].$$

Finally, the bounds $u \le (1-\epsilon) u_\epsilon + O(1)$ and $v \le (1-\epsilon) v_\epsilon + O(1)$ imply

$$\tilde{\tau}_{u_\epsilon} + \tilde{\tau}_{v_\epsilon} \le (1-\epsilon)^{-1} (\tilde{\tau}_u + \tilde{\tau}_v),$$

and the assertion follows. □

6.2 Musţată's summation formula

Given an ideal $I = \langle F_1, \ldots, F_r \rangle \subset \mathcal{O}_x$ and $\gamma > 0$, we denote by $\mathcal{J}(I^\gamma)$ the multiplier ideal of the plurisubharmonic function function $\gamma \log |F|$.

In [15] it was shown that

$$\mathcal{J}((I+J)^\gamma) \subseteq \sum_{\alpha+\beta=\gamma} \mathcal{J}(I^\alpha)\mathcal{J}(J^\beta) \tag{6.2}$$

(as is easy to see, the sum contains finitely many distinct terms).

Its plurisubharmonic counterpart will be about the multiplier ideal of the function $\max\{u, v\}$.

Theorem 6.2 *If $u, w \in \mathrm{PSH}_x$ have asymptotically analytic singularities, then*

$$\mathcal{J}(\max\{u,v\}) \subseteq \sum_{\alpha+\beta=\gamma} \mathcal{J}(\alpha u)\mathcal{J}(\beta v), \quad \forall \gamma < 1.$$

Proof. We will reduce the situation to the one from [15]. In the notation from the proof of Theorem 6.1, we have

$$(1+\epsilon)p\log|F_\epsilon| + O(1) \le Nu \le (1-\epsilon)p\log|F_\epsilon| + O(1), \tag{6.3}$$

$$(1+\epsilon)q\log|H_\epsilon| + O(1) \le Nv \le (1-\epsilon)q\log|H_\epsilon| + O(1) \tag{6.4}$$

with $p, q, N \in \mathbb{Z}_+$. By (6.2),

$$\mathcal{J}((I_\epsilon + J_\epsilon)^\gamma) \subseteq \sum_{\alpha+\beta=\gamma} \mathcal{J}(I_\epsilon^\alpha)\mathcal{J}(J_\epsilon^\beta).$$

Note that $\mathcal{J}(I_\epsilon^\alpha) = \mathcal{J}(\alpha p \log|F_\epsilon|)$ and $\mathcal{J}(J_\epsilon^\beta) = \mathcal{J}(\beta q \log|H_\epsilon|)$. On the other hand, the ideal $I_\epsilon + J_\epsilon$ is generated by the components of F_ϵ^p and H_ϵ^q, so

$$\mathcal{J}((I_\epsilon + J_\epsilon)^\gamma) = \mathcal{J}(\gamma \max\{p\log|F_\epsilon|, q\log|H_\epsilon|\})$$

for any $\gamma > 0$. Take $\gamma = (1-\epsilon)N^{-1}$. By (6.3) and (6.4), we have then

$$\begin{aligned} \mathcal{J}(\max\{u,v\}) &\subseteq \mathcal{J}(\gamma \max\{p\log|F_\epsilon|, q\log|H_\epsilon|\}) = \mathcal{J}((I_\epsilon + J_\epsilon)^\gamma) \\ &\subseteq \sum_{\alpha+\beta=1-\epsilon} \mathcal{J}(\alpha a_\epsilon \log|F_\epsilon|)\mathcal{J}(\beta b_\epsilon \log|H_\epsilon|) \subseteq \sum_{\alpha+\beta=r_\epsilon} \mathcal{J}(\alpha u)\mathcal{J}(\beta u), \end{aligned}$$

where $r_\epsilon = (1-\epsilon)(1+\epsilon)^{-1}$. □

References

[1] S. Boucksom, C. Favre, and M. Jonsson, *Valuations and plurisubharmonic singularities*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 449–494.

[2] U. Cegrell, *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 1, 159–179.

[3] J.-P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geometry **1** (1992), 361–409.

[4] J.-P. Demailly, *Multiplier ideal sheaves and analytic methods in algebraic geometry*, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 1–148, ICTP Lect. Notes, 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[5] J.-P. Demailly, *Estimates on Monge-Ampère operators derived from a local algebra inequality*, preprint at http://arxiv.org/abs/0709.3524v2.

[6] J.-P. Demailly, L. Ein, and R. Lazarsfeld, *Subadditivity property of multiplier ideals*, Michigan Math. J. **48** (2000), 137–156.

[7] L. Ein, R. Lazarsfeld, and K.E. Smith, *Uniform bounds and symbolic powers on smooth varieties*, Invent. Math. **144** (2001), no. 2, 241–252.

[8] L. Ein, R. Lazarsfeld, and K.E. Smith, *Uniform approximation of Abhyankar valuation ideals in smooth function fields*, Amer. J. Math. **125** (2003), no. 2, 409–440.

[9] D. Eisenbud and H.I. Levine, *An algebraic formula for the degree of a C^∞ map germ. With an appendix by Bernard Teissier, "Sur une inégalité à la Minkowski pour les multiplicités"*, Ann. Math. (2) 106 (1977), no. 1, 19–44.

[10] M. Klimek, Pluripotential theory. Oxford University Press, London, 1991.

[11] R. Lazarsfeld, Positivity in Algebraic Geometry, II. Springer-Verlag, Berlin, 2004.

[12] M. Lejeune-Jalabert et B. Teissier, *Clôture intégrale des idéaux et équisingularité*, preprint, 1973. Available at http://arxiv.org/abs/0803.2369

[13] P. Lelong and A. Rashkovskii, *Local indicators for plurisubharmonic functions*, J. Math. Pures Appl. **78** (1999), 233–247.

[14] M. Musţată, On multiplicities of graded sequences of ideals, J. Algebra **256** (2002), 229-249.

[15] M. Musţată, *The multiplier ideals of a sum of ideals*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 205–217.

[16] Nguyen Van Khue and Pham Hoang Hiep, *A comparison principle for the complex Monge-Ampre operator in Cegrell's classes and applications* Trans. Amer. Math. Soc. **361** (2009), no. 10, 5539–5554.

[17] A. Rashkovskii, *Relative types and extremal problems for plurisubharmonic functions*, Int. Math. Res. Not. **2006** (2006), Article ID 76283, 26 p.

[18] A. Rashkovskii, *Tropical analysis of plurisubharmonic singularities*, Tropical and Idempotent Mathematics, 305–315, Contemp. Math., **495**, Amer. Math. Soc., Providence, RI, 2009.

[19] A. Rashkovskii and R. Sigurdsson, *Green functions with singularities along complex spaces*, Internat. J. Math. **16** (2005), no. 4, 333–355.

[20] P. Samuel, *Some asymptotic properties of powers of ideals*, Ann. of Math. (2) **56** (1952). 11–21.

[21] Yang Xing, *Weak convergence of currents*, Math. Z. **260** (2008), no. 2, 253–264.

[22] V.P. Zahariuta, *Spaces of analytic functions and maximal plurisubharmonic functions.* D.Sci. Dissertation, Rostov-on-Don, 1984.

[23] V.P. Zahariuta, *Spaces of analytic functions and Complex Potential Theory*, Linear Topological Spaces and Complex Analysis **1** (1994), 74–146.

Tek/Nat, University of Stavanger, 4036 Stavanger, Norway

E-mail: alexander.rashkovskii@uis.no